Chemical Engineering Journal 111 (2005) 199–204

Chemical Engineering Journal

www.elsevier.com/locate/cej

Physico-mathematical modeling of crossflow filtration

A.S. Ferreira [a], G. Massarani [b,*]

[a] Centro de Pesquisas e Desenvolvimento/Petrobras, Cidade Universitária Q. 7, 21949-900 Rio de Janeiro, RJ, Brazil

[b] PEQ/COPPE, Universidade Federal do Rio de Janeiro, Caixa Postal 689502, 21945-970 Rio de Janeiro, RJ, Brazil

Abstract

A phenomenological model that allows the correlation of pressure fields, filtration rate and cake thickness in the process that results from the axial flow of a suspension in a duct that is provided with a porous wall is presented. The model is based in the motion and the continuity equations for the phases and incorporates the following constitutive information:

- Rheological properties of the suspension and of the fluid that makes up this suspension.
- Rheological properties of the cake including the effects of the compressibility and the relation between the shear stress of mobilization and the normal stress associated to the Coulomb criterion in the stabilization of the thickness of the cake.
- Properties of the filter medium.

The experiments performed with an aqueous suspension of calcium carbonate (average particle diameter in the order of 0.5 μm) confirms the common knowledge that the properties of the cake depend on the mode of filtration, making it of capital importance to perform specific assays for each case that is studied.

Keywords: Physico-mathematical modeling; Crossflow filtration; Non-Newtonian

1. Introduction

Crossflow filtration results from the axial flow of a suspension through a duct which walls are made of filter material, leading to the formation of a deposit on the filter surface and the production of a filtrate that percolates through it. Characteristically, the flow of filtrates drops with time, and may stabilize, in a longer operation, as a consequence of the action of the mechanisms that limit the growth of the cake.

Crossflow filtration is used in the industry for the clarification of effluents and in the concentration of suspensions in a wide range of applications that employ the technology of membranes [1].

The operation of drilling and preparation of inclined oil wells involves essentially the same phenomena that prevail in the crossflow filtration with micro-membranes. It is the axial flow of a non-Newtonian suspension in the annular space between a cylinder in rotation and the formation of petroleum. In this situation, the properties and the thickness of the deposit that is formed allow for the control of the damaging invasion of the drilling fluid in the petroleum formation [2].

This work is restricted to the study of the crossflow filtration that results from the axial flow of a suspension in a hose that was built with the filter medium used in industrial filtration. The modeling is made based on the continuity and motion equations for the phases, and has as its objective to establish the relation between the pressure fields, the flow of filtrate, and the thickness of the cake along the process [3]. The model considers the period of cake growing, with the attendant reduction in the rate of filtration, and the stage in which the thickness of the cake and the flow of filtrate may stabilize in a longer operation, in which the filter works as a thickener.

The constitutive information aggregated to the proposed model are to be established, whenever it is possible, in assays that do not use the results from crossflow filtration itself: rheometry for the survey of the properties of the suspension and of the fluid that makes up this suspension, dead-end filtration for the characterization of the cake and of the filter

* Corresponding author. Tel.: +55 21 2562 8345; fax: +55 21 2562 8300.
E-mail address: gmassa@peq.coppe.ufrj.br (G. Massarani).

1385-8947/$ – see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.cej.2005.02.002

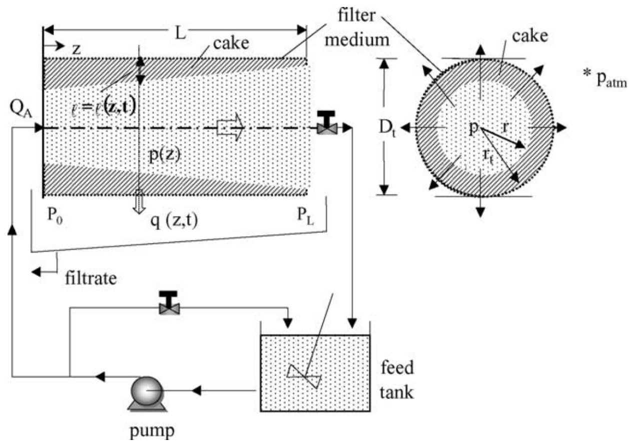

Fig. 1. Rig flow diagram.

medium, assay for establishing the relation between the normal and shear stresses of mobilization for the cake (direct shear test). This last assay is associated to the criterion of Coulomb for the stabilization of the cake.

Taking into consideration the probable differences between the mechanisms of formation of the cake in the modes of filtration, it is recommended that the cake be also characterized by means of assays in the crossflow filtration itself.

2. Physico-mathematical modeling

The modeling of the crossflow filtration that is proposed in this paper corresponds to the situation which scheme is showed in Fig. 1, in which the filtrate is weighed and discarded, and the suspension is returned to the feeding tank. During the stage of cake formation, the concentration of solids is kept constant; the beginning of the thickening of the suspension signals the start of the process of stabilization of the cake.

2.1. Flow of the suspension in the duct with porous wall

The pressure field that is established by the flow of the suspension through the tube may be uncoupled from the process of filtration when the permeability of the porous wall (filter medium + cake) is reduced. This situation widely prevails in the crossflow filtration.

The relation between the pressure drop and the flow rate in the axial flow is given by the following formula [4]:

$$-\frac{\Delta p}{L} = \frac{f V_M^2 \rho_M}{2 D_t} \tag{1}$$

$$\frac{p_0 - p(z)}{p_0 - p_L} = \frac{z}{L} \tag{2}$$

$$f = f\left(\frac{e}{D_t}, Re_M\right), \qquad Re_M = \frac{D_t V_M \rho_M}{\mu_{ef}} \tag{3}$$

$$V_M = \frac{Q_A}{A_t}, \qquad \rho_M = \varepsilon\rho_F + (1-\varepsilon)\rho_s \tag{4}$$

$$\mu_{ef} = \frac{S_M(\lambda^*)}{\lambda^*}, \quad \lambda^* = \frac{6.4 V_M}{D_t} \tag{5}$$

It is important to highlight that these results do not depend on the flow regime and that the effective viscosity μ_{ef} is calculated from the rheology of the suspension, with the knowledge of the characteristic rate of distension λ^*.

2.2. Filtration with cake formation

The initial step of crossflow filtration is characterized by the growth of the cake that is formed next to the filter medium, with the attendant reduction in the rate of filtration. These parameters vary with the position, as the pressure drop in the filtration

$$(\Delta p)_f = p(z) - p_a$$

reduces along the filter tube, as it is indicated by Eq. (2). Being the filtration performed in the circuit that is sketched in Fig. 1, the concentration of the suspension, c, is kept constant during the stage of cake growth.

The equations of conservation of mass for the phases and of the motion of the fluid, in the form of the Darcy equation, lead to the equation for crossflow filtration over a cylindrical surface [5,6]:

$$\frac{dt}{dv} = \frac{\mu_F}{(\Delta p)_f}\left\{\alpha\varepsilon_s\rho_s r_t \ln\frac{r_t}{\left[r_t^2 - \frac{2c\rho_F r_t v}{\varepsilon_s\rho_s}\right]^{1/2}} + R_m\right\} \tag{6}$$

where t is the time of filtration and v is the volume of filtrate per unit area of filtration,

$$v = \frac{1}{2\pi r_t}\frac{dV}{dz} \tag{7}$$

Being the crossflow filtration performed with a suspension that is comprised by particles within a distribution of size, the deposit of these particles is selective along the process that combines axial flow and percolation through the filter medium [3,7]. In other words, the resistivity α and the volumetric fraction of the solids ε_s, averages in each cross-section of the filter, vary not only with the filtration pressure at the location, but also with the structure of the cake that is deposited.

Being the cake compressible, the resistivity α and the volumetric fraction of the solids ε_s depend on the filtration pressure. R_m is the resistance of the filter medium.

The result given by Eq. (5) may be extended for the case in which the liquid that percolates through the cake and the filter medium is non-Newtonian. The effective viscosity may be calculated from the rheology of the liquid with the knowledge of the characteristic rate of distension λ^*, [8]:

$$\mu_{\mathrm{ef}} = \frac{S_{\mathrm{L}}(\lambda^*)}{\lambda^*}, \quad \lambda^* \cong \frac{\left|\frac{dv}{dt}\right|}{k} \tag{8}$$

where k is the permeability of the porous medium.

The thickness of the cake and the total volume of filtrate may be calculated through the expressions:

$$\ell(z,t) = r_t - \left[r_t^2 - \frac{2cr_t v(z,t)}{\varepsilon_{\mathrm{s}}\rho_{\mathrm{s}}}\right]^{1/2} \tag{9}$$

$$V(t) = \pi D_t \int_0^L v(z,t)\,dz \tag{10}$$

2.3. Stabilization of the flow of filtration

The direct shear test, widely used in soil mechanics, allows the correlation, for a bulk material, of the shear stress of mobilization τ and the normal stress σ. The following linear form results for the so-called “Coulomb materials” [9]:

$$\tau = \eta\sigma + C \tag{11}$$

In this equation, η and C are, respectively, the friction coefficient and the cohesion, parameters that also depend on the history of preparation of the sample.

The stabilization of the cake in crossflow filtration is associated to the breaking of the Coulomb equilibrium in the surface of the cake that is in contact with the suspension [7,10],

$$\tau_i \geq \eta\sigma_i + C \tag{12}$$

Considering that only the fluid-dynamic forces are acting on the system [11], the shearing and normal stresses on the surface of the cake are expressed in the form of:

$$\tau_i = \frac{D_{\mathrm{f}}}{4}\left(-\frac{dp}{dz}\right) = \frac{2f\rho_{\mathrm{M}}Q_{\mathrm{A}}^2}{\pi^2 D_{\mathrm{f}}^4} \tag{13}$$

$$\sigma_i = \frac{\mu_{\mathrm{F}}\delta}{k_i}\left(\frac{dv}{dt}\right)_i = \eta^*\left(\frac{dv}{dt}\right)_i \tag{14}$$

where D_{f} is the free diameter of the tube for the axial flow of the suspension

$$D_{\mathrm{f}} = D_t - 2\ell \tag{15}$$

and δ is the thickness associated to the average diameter of the particles, D_{p}. The permeability k_i is related to the resistivity and to the volumetric fraction of solids in the surface of the cake

$$k_i = \frac{1}{\rho_{\mathrm{s}}\varepsilon_{\mathrm{s}i}\alpha_i} \tag{16}$$

Eq. (14) results from the resistive force (per unit volume of the particulate system) that the fluid exerts on the porous matrix during the Darcyan flow [8],

$$\boldsymbol{m} = \frac{\mu_{\mathrm{F}}}{k}\boldsymbol{q} \tag{17}$$

In summary, the growth in thickness of the cake in crossflow filtration, and the attendant reduction in the rate of filtration, leads to the breaking of the Coulomb equilibrium by the reduction of D_{f}, increase in the shear stress and reduction in the normal stress that act on the surface of the cake.

2.4. Simplified theory for the crossflow filtration

In the following analysis, it is considered that the average properties of the cake are a function only of the pressure of filtration, and that the percolating fluid is Newtonian. The integration of Eq. (6) leads to the results

$$t = \frac{\mu_{\mathrm{F}}}{(\Delta p)_{\mathrm{f}}}\left\{\alpha\varepsilon_{\mathrm{s}}\rho_{\mathrm{s}}r_t\sum_{n=1}^{\infty}\frac{b^n}{2n(n+1)r_t^{2n}}v^n + R_{\mathrm{m}}\right\}v,$$
$$b = \frac{2c\rho_{\mathrm{F}}r_t}{\varepsilon_{\mathrm{s}}\rho_{\mathrm{s}}}, \quad (\Delta p)_{\mathrm{f}} = \varphi_1(z), \quad \alpha = \alpha\left\lfloor(\Delta p)_{\mathrm{f}}\right\rfloor = \varphi_2(z),$$
$$\varepsilon_{\mathrm{s}} = \varepsilon_{\mathrm{s}}\left\lfloor(\Delta p)_{\mathrm{f}}\right\rfloor = \varphi_3(z) \tag{18}$$

$$\ell(z,t) = r_t - \left[r_t^2 - \frac{2cr_t v(z,t)}{\varepsilon_{\mathrm{s}}\rho_{\mathrm{s}}}\right]^{1/2} \tag{9}$$

$$V(t) = \pi D_t \int_0^L v(z,t)\,dz \tag{10}$$

The next step is related to the estimate of the flow rate of filtration when the operation reaches steady state. The cake that results from the crossflow filtration probably has properties that are distinct from the sample that was prepared for the performance of the shear test [12]. It is important to highlight that it is not evident the association of the parameter δ to an average diameter of the particle, in the calculation of the normal stress that acts on the surface of the cake, Eq. (14). It results from these considerations that it seems more significant to estimate the rheological parameters of the cake by means of assays performed in the crossflow filtration itself.

The stabilization of the thickness of the cake does not to take place simultaneously in all points of the system. With the objective of making the derivation even simpler, it is proposed that the development of the volume of filtrate with the time of filtration be estimated based on the “method of the two asymptotes” of Churchill [13], by the heuristic combination of the results during the transient and stationary periods

$$\frac{t}{V} = \left[\left(\frac{t}{V}\right)_t^{-n} = \left(\frac{dV}{dt}\right)_\infty^n\right]^{-1/n} \tag{19}$$

In this last equation, n is a parameter of adjustment to the experimental data, which by inspection seems to assume a value that is close to 2. $(t/V)_t$ may be calculated from equation

$$\left(\frac{t}{V}\right)_t = \frac{t}{\pi D_t \int_0^L v(z,t)\,\mathrm{d}z} \tag{20}$$

recovering the function $v(z, t)$ of Eq. (18).

The flow rate of filtration when the process reaches steady state, $(\mathrm{d}V/\mathrm{d}t)_\infty$, and the equilibrium thickness, ℓ_∞, may be estimated solving the system of equations

$$\frac{D_t - 2\ell_\infty}{4}\left(-\frac{\Delta p}{L}\right) = \frac{\eta^*}{A_\mathrm{t}}\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_\infty + C,$$

$$\ell_\infty = \frac{D_t}{2}\left\{1 - \exp\left[\frac{\Omega_1(\Delta\bar{p})_\mathrm{f}}{\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_\infty} - \Omega_2\right]\right\},$$

$$\Omega_1 = \frac{2\pi L}{\rho_\mathrm{s}\alpha\varepsilon_\mathrm{s}\mu_\mathrm{F}},\quad \Omega_2 = \frac{2R_\mathrm{m}}{D_t\rho_\mathrm{s}\alpha\varepsilon_\mathrm{s}} \tag{21}$$

Eq. (21) is the result of the integration of Darcy equation for the flow of a fluid in concentric cylindrical shells: the filter medium and the cake considered as having a constant thickness, ℓ_∞.

3. Experiments and results

The experiments had the objective of characterizing the properties of the cake that resulted from the crossflow filtration and the comparison of the results with those obtained in easier conditions: in dead-end filtration and using the direct shear test.

The experiments were performed in the system that is sketched in Fig. 1, with aqueous suspensions of pharmaceutical calcium carbonate (Herzog), in a range of concentrations between 2 and 5% (w/w). The filter hose, with 3 cm in diameter and in the lengths of 31 and 88 cm, was built in polyester fabric that is used as filter medium in industrial processes.

The physical properties of the system that was studied are gathered in Table 1. The suspension of calcium carbonate exhibited a Newtonian behavior. Assays performed with dead-end filtration showed that the cake that was formed may be considered as being incompressible, and the direct shear test indicated that the cake has an internal friction angle of the order of 40°, as well as nil cohesion.

The experiments consisted in the direct measurement of the development of the volume of filtrate as a function of the time of filtration, operating the filter with a controlled feed pressure, p^0. The feed flow rate Q_A and the concentration of solids in the suspension C were maintained almost constant along the cycle of each experiment. The beginning of the counting of the time, the initial time, was made in an arbitrary way, when the filtrate presented itself clear, which indicated the formation of some deposit of particles on the filter surface, a deposit which was macroscopically taken into consideration as a new value for the resistance R_{m^*}, that also included the cake that was deposited initially. R_{m^*} is to be calculated for each experimental cycle.

Table 2 contains the specifications related to the 12 runs of crossflow filtration that were performed in the Laboratory of Particulate Systems at COPPE/UFRJ. It is seen that in all cases the pressure drop in the axial flow is small relative to the feed pressure in the filter, indicating that the filtration pressure along the filter may be considered constant,

$$(\Delta p)_\mathrm{f} = p_0 = p_\mathrm{atm}$$

Fig. 2 shows four typical runs that confirm the common knowledge that, for a same volume of filtrate, the rate of

Table 2
Experiments on crossflow filtration

Run	p_0 (atm)	Q_A (l/min)	c_0 (w/w)	$p_0 - p_\mathrm{L}$ (kPa)	Time of a cycle (min)
1	2.0	22.7	0.018	0.177	10
2	3.1	53.6	0.017	0.934	10
3	1.8	28.4	0.022	0.273	10
4	1.6	20.9	0.017	0.150	10
5	1.7	24.9	0.018	0.205	10
6	3.2	50.3	0.033	1.10	60
7	2.2	35.2	0.033	0.412	60
8	2.0	32.9	0.042	0.363	60
9	2.0	11.6	0.030	0.043	60
10	2.0	30.5	0.045	0.110	120
11	1.5	9.1	0.032	0.011	120
12	2.0	18.7	0.023	0.041	120

D_t = 3 cm; L = 88 cm (runs 1–9), L = 31 cm (runs 10–12).

Table 1
Characterization of the system

Property	Results	Experimental
Particle size distribution of the $CaCO_3$ used in suspension (ρ_s = 2.91 g/cm^3)	$D(v, 0.1) = 0.1$ µm, $D(v, 0.5) = 0.30$ µm, $D(v, 0.9) = 7.16$ µm, $D[3, 2] = 0.24$ µm, $D[4, 3] = 1.79$ µm	Malvern mastersizer
Rheology of the suspension concentration up to 5% (w/w)	Viscosity of water	Capillary viscometer
Resistance of the filter medium (polyester)	$R_\mathrm{m} = (2.1 \pm 0.84) \times 10^9$ cm^{-1} (10 experiments)	Dead-end filtration
Porosity and resistivity of the cake	$0.25 < (\Delta p)_\mathrm{f} < 3$ atm, $\varepsilon = 0.74 \pm 0.05$ (59 experiments), $\alpha = (1.35 \pm 0.27) \times 10^{10}$ cm/g (10 runs)	Dead-end filtration
Rheology of the cake, $\varepsilon = 0.62$, $\sigma < 8.5$ kPa	$\tau = 0.85\sigma + 0$ (3 points)	Shear box test (Ronald Top)

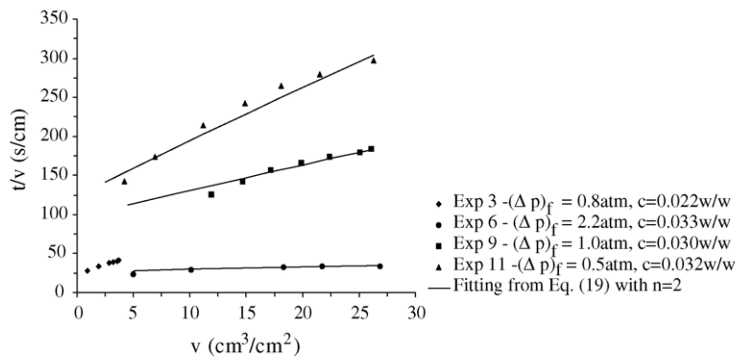

Fig. 2. Experimental measurements on crossflow filtration.

Table 3
Crossflow filtration: results

Property	Dead-end filtration and shear box test	Crossflow filtration
Porosity and resistivity of the cake	$0.25 < (\Delta p)_f < 3$ atm, $\varepsilon = 0.74$, $\alpha = 1.35 \times 10^{10}$ cm/g	$0.5 < (\Delta p)_f < 2$ atm, $\varepsilon = 0.65$[a], $\alpha = (2.6 \pm 0.26) \times 10^{10}$ cm/g[b] (runs 1–5)
Rheology of the cake	$\varepsilon = 0.62$, $\tau = 0.85\sigma$ ($\sigma < 85$ kPa)	$\varepsilon = 0.65$, $\tau = 2.67 \times 10^3 \frac{dv}{dt}$ (CGS units)[c], $\delta = 0.11$ µm (Fig. 3, runs 6–12)

[a] The porosity of the cake was evaluated from Kozeny–Carman equation: $\frac{\alpha_1}{\alpha_2} = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^3 \frac{1-\varepsilon_1}{1-\varepsilon_2}$.

[b] The resistivity of the cake was calculated from Eq. (20): $\frac{t}{V} = \frac{\mu_F}{A_t(\Delta p)_f}\left[\frac{\alpha\rho_F c V}{2A_t} + R_m\right]$.

[c] The mobility shear stress and the normal stress were calculated from Eqs. (19) ($n = 2$), (20), (21) and (14):

$$\left\{\begin{array}{l} \dfrac{t}{V} = \left[\left(\dfrac{t}{V}\right)_t^{-2} + \left(\dfrac{dV}{dt}\right)_\infty^2\right]^{1/2} \\ \hline \left(\dfrac{t}{V}\right)_t = \dfrac{\mu_F}{A_t(\Delta p)_f}\left[\dfrac{\alpha\rho_F c V}{2A_t} + R_m^*\right] \end{array}\right. \to R_m^*, \left(\frac{dV}{dt}\right)_\infty$$

$$\ell_\infty = \frac{D_t}{2}\left\{1 - \exp\left[\frac{\Omega_1(\Delta p)_f}{\left(\dfrac{dV}{dt}\right)_\infty} - \Omega_2\right]\right\} \to \ell_\infty$$

$$\frac{D_t - 2\ell_\infty}{4}\left(-\frac{\Delta p}{L}\right) = \eta^*\left(\frac{dv}{dt}\right)_\infty \to \eta^*$$

filtration increases with the increase in the filtration drop and with the reduction in the concentration of solids in the suspension. The representation of the results as t/V against V is very usual in the analysis of dead-end filtration under a constant pressure drop. The linear dependence between these variables shows the plane filtration and the deviations indicate, in the increasing direction of t/V, that the process gradually moves towards stabilization.

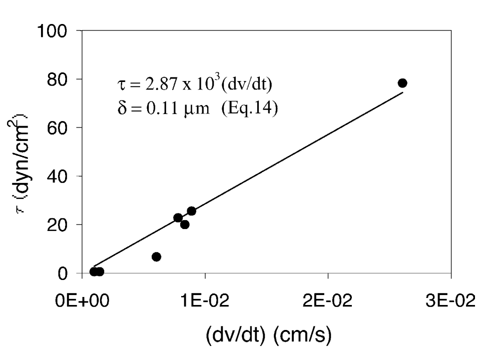

Fig. 3. Permeate velocity vs. shear stress.

The comparison between the results of the characterization of the cake obtained from the dead-end filtration, direct shear test and crossflow filtration is done in Table 3 (Fig. 3). It is important to highlight that the resistivity of the cake obtained in the crossflow filtration is larger than that which is reached with the dead-end filtration, as it is pointed by Mikulásek et al. [14], even though the reverse situation seems more common [12].

4. Conclusions

It is presented a phenomenological model that allows the correlation of the pressure field, rate of filtration and thickness of the cake in the process that results from the axial flow of a suspension in a duct provided with a porous wall: Eqs. (2), (6), (9) and (10) and the restrictive condition to the growth of the thickness of the cake (12). The model is based in the motion and continuity equations for the phases, and incorporates the following constitutive information:

- Rheological properties of the suspension and of the fluid that makes up this suspension.

- Rheological properties of the cake including the compressibility data and the relation between the shear stress of mobilization and the normal stress associated to the Coulomb criterion in the stabilization of the thickness of the cake.
- Properties of the filter medium.

A simplified model that results from the heuristic combination of the estimates relative to the stage of cake growth and when the process reaches the established regime is also suggested.

The experiments performed with an aqueous suspension of pharmaceutical calcium carbonate showed that the properties of the cake depend on the mode of filtration, what confirms the common knowledge and makes it indispensable to perform the specific characterization assays.

The study of crossflow filtration would greatly benefit from a better knowledge of the local properties of the cake that is formed by the selective deposit of the solid particles along the filter medium. In this sense, it is proposed the performance of experiments that allow the measurement of the volume of filtrate in sections of the filter surface.

Acknowledgement

The preparation of this paper and its presentation at the conference were made possible through the support by the Conselho Nacional de Desenvolvimento Científico e Tecnológico do Brasil (CNPq), grant #300113/92.7.

References

[1] S. Ripperger, J. Altmann, Sep. Pur. Technol. 26 (2002) 19–31.
[2] A.T. Bourgoyne, K.K. Willhein, M.E. Chenevert, F.S. Young Jr., Applied Drilling Engineering, Society of Petroleum Engineers, USA, 1991.
[3] G. Belfort, N. Nagata, Desalination 53 (1985) 57–79.
[4] M.C. Ferreira, J.T. Freire, G. Massarani, Powder Technol. 108 (2000) 46–54.
[5] F.M. Tiller, C.S. Yeh, AIChE J. 31 (1985) 1241–1248.
[6] G. Massarani, A.M. Silveira, Proceedings of the XIII Brazilian Congress on Particulate Systems, Sao Paulo, Brazil, 1985, pp. 1–8.
[7] N.J. Blake, I.W. Cumming, M. Streat, J. Membr. Sci. 58 (1992) 205–216.
[8] G. Massarani, A. Silva Telles, J. Porous Media 4 (2001) 297–307.
[9] R.L. Brown, J.C. Richards, Principles of Powder Mechanics: Essays on Packing and Flow of Powders and Bulk Solids, Pergamon Press, Oxford, UK, 1970.
[10] Y.K. Benkahla, A. Ould-Dris, M.Y. Jaffrin, D. Si-Hansen, J. Membr. Sci. 98 (1995) 107–117.
[11] S.J. Hwang, D.J. Chang, C.H. Chen, Chem. Eng. J. 61 (1996) 171–178.
[12] Y. Xu-Jiang, J. Dodds, D. Leclerc, Filtr. Separat. (1995) 795–798.
[13] S.W. Churchill, Rev. Latino Am. Transf. Cal. Mat. 7 (1983) 207–229.
[14] P. Mikulásek, R.J. Wakerman, J.Q. Marchant, Chem. Eng. J. 69 (1998) 53–61.